# THE NTH DIAMETER OF A REAL INTERVAL

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## Abstract

A formula for the nth diameter of a real interval is proved.

In [1, p. 229] Fekete defines for an arbitrary compact subset $E$ of the complex plane and for each natural number $n \geq 2$ the $n$th diameter of $E$ as

$$d_n = \sqrt[n(n-1)]{D_n}, \tag{1}$$

where $D_n$ is the maximal value of the discriminant of $n$ points in $E$. He shows that the sequence $d_2, d_3, \ldots$ is decreasing and converges towards a non-negative number, the transfinite diameter $d$ of $E$.

The value of $d$ for a certain set $E$ is often easy to calculate, using methods from complex function theory. But also the sequence $d_n$ is of interest, for instance (if $d < 1$ and $E$ is symmetrical with respect to the real axis) for obtaining simple estimates of the highest degree $N$ of a monic polynomial $P$ with integer coefficients and the property that its zeros are simple and all lie in $E$: We must have $N < n_{min}$, where $n_{min}$ is the lowest value of $n$ for which the $n$th diameter $d_n$ of $E$ is less than one(see [1, p. 243]).

We shall consider the particular case where $E$ is a real interval $(a, b)$.

To calculate the $n$th diameter $d_n$ of this interval for a given value of $n \geq 2$ we use the definition,

$$d_n = \sqrt[n(n-1)]{\max\{\prod_{j=1}^{n-1} \prod_{k=j+1}^{n} (x_k - x_j)^2\}}, \tag{2}$$

where the parameters $x_1, x_2, \ldots, x_n$ satisfy the inequalities

$$a \leq x_1 < x_2 < \cdots < x_n \leq b. \tag{3}$$

Thus, we simply have to maximize the discriminant of the numbers $x_1, x_2, \ldots, x_n$ subject only to the restrictions (3).

It is easily seen that for a given value of $n$ the ratio $d_n/d$ does not depend on the interval, and so we shall make the standard choice $a = -1$ and $b = 1$.

Clearly, we must have $x_1 = -1$ and $x_n = 1$. Taking logarithmic partial derivatives of the discriminant with respect to the parameters $x_2, x_3, \ldots, x_{n-1}$ we find the equations

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k} = 0 \qquad \text{for } j = 2, 3, \ldots, n-1. \tag{4}$$

Introducing the polynomial

$$P_n(x) = \prod_{k=1}^{n} (x - x_k), \tag{5}$$

we have the identity

$$P_n'(x) = P_n(x) \sum_{k=1}^{n} \frac{1}{x - x_k},$$

implying

$$P_n'(x) - \frac{P_n(x)}{x - x_j} = P_n(x) \sum_{k \neq j} \frac{1}{x - x_k},$$

which by differentiation at $x_j$ (use the expansion of $P_n(x)$ at this point) yields

$$\frac{1}{2}P_n''(x_j) = P_n'(x_j)\sum_{k\neq j}\frac{1}{x_j - x_k},$$

which shows that $x_2, x_3, \ldots, x_{n-1}$ are precisely the zeros of $P_n''(x)$. So the polynomial $P_n$ must satisfy the differential equation

$$n(n-1)P_n(x) = (x^2-1)P_n''(x). \tag{6}$$

Here we have compared the leading terms to get the factor $n(n-1)$.

Comparing in turn the other terms of decreasing degree, we see that the monic polynomial $P_n$ is, in fact, uniquely determined as a solution to (6). It has the form

$$P_n(x) = (x^2-1)\frac{1}{k_{n-2}}P_{n-2}^{(1,1)}(x) \tag{7}$$

in the notation of [3, pp. 272–274].

The number $k_{n-2}$ is the leading coefficient of the polynomial $P_{n-2}^{(1,1)}$ as defined for instance in [3], where one can also find a description of the properties of these polynomials needed in the following.

For our applications it is more natural to define the Jacobi polynomials as monic, and we shall do so in the sequel.

Note also that nominally we then have $P_n = P_n^{(-1,-1)}$ (only defined for $n \geq 2$).

Since the transfinite diameter $d$ equals $1/2$ for our chosen interval, the quantity we look for is

$$\frac{d_n}{d} = 2D_n^{1/(n(n-1))}, \tag{8}$$

where $D_n$ is the discriminant of the polynomial $P_n$.

As shown below, it is possible to get a simple explicit expression for $D_n$ or, more generally, for the discriminant of the Jacobi polynomial $P_n^{(\alpha,\beta)}$. We have the

**Lemma.** *The discriminant of the Jacobi polynomial* $P_n^{(\alpha,\beta)}$ *is*

$$\begin{aligned} D_n^{(\alpha,\beta)} &= \prod_{k=2}^{n}\frac{4^{k-1}k^k(\alpha+k)^{k-1}(\beta+k)^{k-1}(\alpha+\beta+k)^{k-2}}{(\alpha+\beta+2k)^{2k-2}(\alpha+\beta+2k-1)^{2k-3}} \\ &= 2^{n(n-1)}\prod_{k=1}^{n}\frac{k^k(\alpha+k)^{k-1}(\beta+k)^{k-1}}{(\alpha+\beta+k+n)^{n+k-2}}. \end{aligned} \tag{9}$$

*Remark.*

This expression can also be obtained by combining the formulae (4.21.6) and (6.71.5) of [4].

*Proof.*

Notation: Since $\alpha$ and $\beta$ are fixed during this demonstration, we shall denote $P_n^{(\alpha,\beta)}$ simply by $P_n$.

We shall need the differential equation

$$(x^2-1)P_n''(x) = n(\alpha+\beta+n+1)P_n(x) - (x(\alpha+\beta+2)+\alpha-\beta)P_n'(x) \tag{10}$$

and the recurrence relation

$$P_n(x) = (x+B_n)P_{n-1}(x) - C_nP_{n-2}(x), \tag{11}$$

where

$$B_n = \frac{\alpha^2-\beta^2}{(\alpha+\beta+2n)(\alpha+\beta+2n-2)}. \tag{12}$$

About $C_n$ we need only know that it exists.

The second derivative of (11) is

$$P_n''(x) = (x+B_n)P_{n-1}''(x) - C_nP_{n-2}''(x) + 2P_{n-1}'(x). \tag{13}$$

Multiplying with $(x^2-1)$ and using (10) we obtain

$$\begin{aligned} &n(\alpha+\beta+n+1)P_n(x) - (x(\alpha+\beta+2)+\alpha-\beta)P_n'(x) \\ &= (x+B_n)((n-1)(\alpha+\beta+n)P_{n-1}(x) - (x(\alpha+\beta+2)+\alpha-\beta)P_{n-1}'(x)) \\ &- C_n((n-2)(\alpha+\beta+n-1)P_{n-2}(x) - (x(\alpha+\beta+2)+\alpha-\beta)P_{n-2}'(x)) \\ &+ 2(x^2-1)P_{n-1}'(x). \end{aligned} \tag{13}$$

Using (11) to eliminate $C_n$ we can simplify to

$$\begin{aligned} &2(\alpha+\beta+2n-1)P_n(x) \\ &= (2x(\alpha+\beta+n)+\alpha-\beta+B_n(\alpha+\beta+2n-2))P_{n-1}(x) + 2(x^2-1)P_{n-1}'(x), \end{aligned} \tag{14}$$

and, by means of (12), to

$$(\alpha+\beta+2n-1)P_n(x) = (\alpha+\beta+n)\left(x+\frac{\alpha-\beta}{\alpha+\beta+2n}\right)P_{n-1}(x) + (x^2-1)P_{n-1}'(x). \tag{15}$$

Differentiating and using (10) results in

$$(\alpha+\beta+2n-1)P_n'(x) = n(\alpha+\beta+n)P_{n-1}(x) + n\left(x+\frac{\beta-\alpha}{\alpha+\beta+2n}\right)P_{n-1}'(x). \tag{16}$$

Solving the pair ((15), (16)) with respect to $P_{n-1}(x)$ and $P_{n-1}'(x)$, we obtain

$$\begin{aligned} P_{n-1}(x) = P_n(x)&\frac{(\alpha+\beta+2n)(\alpha+\beta+2n-1)((\alpha+\beta+2n)x+\beta-\alpha)}{4(\alpha+\beta+n)(\alpha+n)(\beta+n)} \\ + P_n'(x)&\frac{(1-x^2)(\alpha+\beta+2n)^2(\alpha+\beta+2n-1)}{4n(\alpha+\beta+n)(\alpha+n)(\beta+n)} \end{aligned} \tag{17}$$

and

$$P'_{n-1}(x) = -P_n(x)\frac{(\alpha+\beta+2n)^2(\alpha+\beta+2n-1)}{4(\alpha+n)(\beta+n)} + P'_n(x)\frac{(\alpha+\beta+2n)(\alpha+\beta+2n-1)((\alpha+\beta+2n)x+\alpha-\beta)}{4n(\alpha+n)(\beta+n)}. \tag{18}$$

To derive an explicit expression for the ratio $D_n^{(\alpha,\beta)}/D_{n-1}^{(\alpha,\beta)}$ we note that for the monic polynomials $P_n$ (with zeros $x_1,\dots,x_n$) and $P_{n-1}$ (with zeros $y_1,\dots,y_{n-1}$) the absolute value of the resultant is

$$\begin{aligned} |R(P_n,P_{n-1})| &= \prod_{j=1}^{n}\prod_{k=1}^{n-1}|x_j-y_k| \\ &= \prod_{j=1}^{n}|P_{n-1}(x_j)| = \prod_{k=1}^{n-1}|P_n(y_k)|, \end{aligned} \tag{19}$$

while the discriminant of $P_n$ satisfies

$$D_n^{(\alpha,\beta)} = \prod_{j=1}^{n}|P'_n(x_j)|. \tag{20}$$

Substituting successively $x=y_1,\dots,y_{n-1}$ in (15) and multiplying the results, we obtain

$$(\alpha+\beta+2n-1)^{n-1}|R(P_n,P_{n-1})| = D_{n-1}^{(\alpha,\beta)}\prod_{k=1}^{n-1}(1-y_k^2), \tag{21}$$

while putting $x=x_1,\dots,x_n$, successively, in (17) and multiplying gives

$$|R(P_n,P_{n-1})| = D_n^{(\alpha,\beta)}\left(\frac{(\alpha+\beta+2n)^2(\alpha+\beta+2n-1)}{4n(\alpha+\beta+n)(\alpha+n)(\beta+n)}\right)^n\prod_{j=1}^{n}(1-x_j^2). \tag{22}$$

Using

$$\prod_{j=1}^{n}(1-x_j^2) = |P_n(1)P_n(-1)| = \binom{n+\beta}{n}\binom{n+\alpha}{n}2^{2n}\Big/\binom{2n+\alpha+\beta}{n}^2, \tag{23}$$

we finally deduce

$$D_n^{(\alpha,\beta)}/D_{n-1}^{(\alpha,\beta)} = \frac{4^{n-1}n^n(\alpha+n)^{n-1}(\beta+n)^{n-1}(\alpha+\beta+n)^{n-2}}{(\alpha+\beta+2n)^{2n-2}(\alpha+\beta+2n-1)^{2n-3}}, \tag{24}$$

which, together with $D_1^{(\alpha,\beta)}=1$, yields (9). □

The lemma cannot be used directly for $\alpha=\beta=-1$, althhough it does give the right result. It is safer, in this case, to use (24) only for $n\geq 3$ and note that $D_2=4$. Anyway, the result is

$$D_n = 4\prod_{k=2}^{n-1}\frac{(k+1)^{k+1}(k-1)^{k-1}}{(2k-1)^{2k-1}}. \tag{25}$$

from which the ratio $d_n/d$ can be calculated using (8).

Numerical experiments seem to show that the value $n_{min}$ corresponding to a given interval-length $d_2$ (see the beginning of this note) is somewhat too large, in particular for $d_2$ close to 4, and probably does not have the correct asymptotic behaviour in this limit.

For instance, $d_2/d_8 = 2.669$, app., while the polynomial

$$P_8(x) = (x^3 - x^2 - 2x + 1)(x^2 - x - 1)(x + 1)x(x - 1)$$

has a span (difference between maximal and minimal zero) of 3.049, app.

And $d_2/d_{20} = 3.288$, app., while the polynomial

$$\begin{aligned} P_{20}(x) =&(x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1)(x^3 - x^2 - 2x + 1)(x^3 - 3x - 1) \\ &(x^2 - 2)(x^2 - x - 1)(x^2 + x - 1)(x + 1)x(x - 1) \end{aligned}$$

has span 3.6015, app.

There are two reasons for this discrepancy:

1) The zeros of our integer polynomials are not distributed in the span so as to give maximal value of the discriminant.

2) The values of the discriminants are integers, but often (especially for large values of the degree) much greater than one.

At least in the cases studied, the second effect is much more important than the first one. For instance, for $n = 8$ we find the discriminant equal to 980, which with optimal distribution of zeros corresponds to an interval of length 3.018, app. (rather close to the span of 3.049 actually found), while, for $n = 20$, the discriminant equals 26777147443200, corresponding to an interval of length 3.5665, app. (the span was 3.6015, app.).

Thus it is essential, if more accurate determinations of maximal degrees $N$ are to be made, that good lower bounds for the obtainable values of the discriminant are found.

Some elementary results should be mentioned here:

If $n = 2$, we may have a discriminant equal to one: $P(x) = x(x-1)$, for instance.

If $n = 3$, one can use the well known formula for the discriminant and show that the discriminant cannot be congruent to 2 modulo 4. A more involved argument (see below) shows that there are no monic polynomials in $\mathbb{Z}[x]$ with the value 1 for the discriminant. Thus, for monic polynomials of degree 3, the lowest value for the discriminant is 3; in fact, all such polynomials are equivalent (in the sense of Robinson: any one can be obtained from any other by a tranformation $x \mapsto \pm x + t$, where $t \in \mathbb{Z}$) with $P(x) = x^3 + x^2 + x$ (this can be seen by means of the method described below), so that, if we further require that all zeros be real, we have 4 as the minimal value of the discriminant (example: $P(x) = x(x - 1)(x - 2)$).

If $n = 4$, we have a similar situation as for $n = 3$: The discriminant has lowest positive value equal to 3 (thanks to the cubic resolvent the proofs are similar to but somewhat more complicated than those indicated for $n = 3$), realized for example by $P(x) = x(x - 1)(x^2 - x + 1)$. If we want all zeros real, the lowest positive value of the discriminant is probably 5, with $P(x) = x(x - 1)(x^2 - x - 1)$ as an example.

And now the proof that the discriminant for a monic polynomial in $\mathbb{Z}[x]$ of degree 3 cannot be equal to 1:

Let the polynomial be

$$P(x) = x^3 + cx^2 + bx + a. \tag{26}$$

Its discriminant is

$$D = |b^2c^2 - 4b^3 - 4ac^3 - 27a^2 + 18abc|. \tag{27}$$

If $D = 1$, an argument modulo 4 shows that we must have

$$b^2c^2 - 4b^3 - 4ac^3 - 27a^2 + 18abc - 1 = 0. \tag{28}$$

An argument modolo 9 shows that $c$ is not divisible by 3. Solving (28) for $a$ we obtain

$$a = \frac{9bc - 2c^3 \pm \sqrt{(9bc - 2c^3)^2 - 27(1 + 4b^3 - b^2c^2)}}{27}. \tag{29}$$

Assume that we can find integers $b$ an $c$ with $c$ not divisible by 3, such that

$$R = \sqrt{(9bc - 2c^3)^2 - 27(1 + 4b^3 - b^2c^2)} \tag{30}$$

is real (and thus an integer).

The two possible numerators of $a$ in (29) cannot both be divisible by 3 (otherwise their difference, and thus $c$, would be divisible by 3). So, the condition that one of these numerators be divisible by 27 is equivalent to the condition that their product should be divisible by 27, and this is obviously true. "All" we have to do is to solve (30) under the conditions indicated above. We rewrite (30) as the diophantine equation

$$R^2 = 4t^3 - 27, \tag{31}$$

where

$$t = c^2 - 3b. \tag{32}$$

The condition that $c$ is not divisible by 3 translates to the requirement that $t$ must not be divisible by 3. If (31) is solvable in integers $R$ and $t$ satisfying this condition, $t$ must be congruent to 1 moduli 3, and so $c$ of (32) can be chosen arbitrarily (though not divisible by 3), and the $b$ satisfying (32) must then be an integer.

But the only solution of (31) in integers is (see [2, page 247, Theorem 5])

$$(R, t) = (9, 3), \tag{33}$$

and so we cannot have $D = 1$ for the polynomial (33).

An experimental result: It seems that with increasing degree the numeric difference between the minimal values of the discriminant in the general (monic) case, and the case where also all zeros are required to be real, increases markedly. And, of course, with increasing degree also the degrees of some of the fields involved increase, and so the minimal discriminant will approach infinity when the degree of the polynomial increases indefinitely.

*Remark.* This paper was concluded in 1997, except for a few remarks added in 2009.

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## References


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