

Poiseuille Flow (Exercise 3)

33241 Theory of Lab-on-a-Chip Systems

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3.1 Poiseuille flow in a rectangular pipe – the structure of the velocity field

In a rectangular pipe, with the dimensions depicted in figure 1, there is a flow in the x direction with $-\Delta p$: $p(0) = p_0 + \Delta p$ and $p(L) = p_0$. Constant density $\varrho(x, y, z, t) = \varrho_0$ and the no-slip boundary condition are assumed fulfilled.

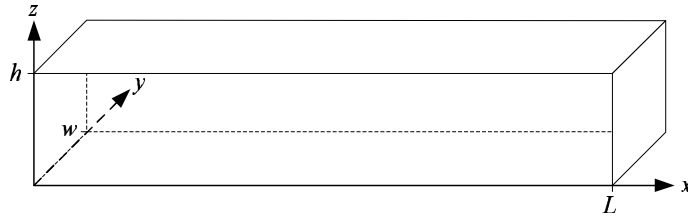


Figure 1: Geometry of the rectangular pipe.

Clearly the flow will only have a velocity field \mathbf{U} acting in the x direction. Furthermore u_x can only be a function of the y and z since the system is translational invariant along the x axis. To summarise \mathbf{U} must be on the form:

$$\mathbf{U}(x, y, z) = (u_x(y, z), 0, 0) \quad (1)$$

The continuity equation must of course be fulfilled to ensure mass conservation. This is checked:

$$\begin{aligned} \frac{\partial}{\partial t} \varrho(x, y, z, t) &= -\nabla \cdot \mathbf{J} = -\nabla \cdot (\varrho(x, y, z, t) \mathbf{U}) \quad \wedge \quad \varrho(x, y, z, t) = \varrho_0 \\ \Rightarrow \quad 0 &= \nabla \cdot \mathbf{U} \\ \nabla \cdot \mathbf{U} &= \partial_x u_x(y, z) + \partial_y 0 + \partial_z 0 \quad \text{where} \quad \partial_i \equiv \frac{\partial}{\partial_i} \\ &= 0 \end{aligned} \quad (2)$$

3.2 Poiseuille flow in a rectangular pipe – Fourier transformation

With $\varrho = \varrho_0$, no acting body forces, and the structure (1) of \mathbf{U} the Navier-Stokes equation reduces to:

$$\begin{aligned}\varrho \frac{D}{Dt} \mathbf{U} &= \varrho \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \mathbf{U} \\ &= -\nabla P + \eta \nabla^2 \mathbf{U} + \mathbf{f} \\ \Rightarrow \nabla^2 u_x(y, z) &= \frac{\nabla P}{\eta} \\ &= \frac{p(L) - p(0)}{\eta L} \\ &= \frac{-\Delta p}{\eta L}\end{aligned}\tag{3}$$

The traditional time domain Fourier series is given by:

$$\begin{aligned}f(t) &\simeq \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \\ a_n &= \frac{2}{T} \int_0^T f(t) \cos(n\omega t) dt \\ b_n &= \frac{2}{T} \int_0^T f(t) \sin(n\omega t) dt\end{aligned}\tag{4}$$

This series is easily transformed to the space domain by the substitution $n\omega t = n\frac{\pi}{w}y$ or $n\omega t = n\frac{\pi}{h}z$. The two-dimensional space domain Fourier series of $u_x(y, z)$ is:

$$u_x(y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin\left(n\frac{\pi}{w}y\right) \sin\left(m\frac{\pi}{h}z\right)\tag{5}$$

where u_{nm} correlates the the y and z dependence, so that in general $u_x(y, z) \neq u_x(y)u_x(z)$. That $u_x(y, z)$ is only given by sine functions is due to the boundary conditions. However, this does not explain why cross products of sines and cosines does not exist. That (5) is a solution can, however, be shown to be true by showing that it obeys the boundary conditions, and finding u_{nm} so that (3) is fulfilled. The boundary conditions are obeyed since:

$$\begin{aligned}u_x(0, z) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin(0) \sin\left(m\frac{\pi}{h}z\right) = 0 \\ u_x(w, z) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin(n\pi) \sin\left(m\frac{\pi}{h}z\right) = 0 \\ u_x(y, 0) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin\left(n\frac{\pi}{w}y\right) \sin(0) = 0 \\ u_x(y, h) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin\left(n\frac{\pi}{w}y\right) \sin(m\pi) = 0\end{aligned}$$

One way of determining u_{nm} is by making an appropriate Fourier series of the right hand side of (3) and compare it to the left. The Fourier series of 1 is:

$$\begin{aligned} a_n &= \frac{2}{w} \int_0^w \cos\left(n\frac{\pi}{w}y\right) dt = \frac{2}{n\pi} \left[\sin\left(n\frac{\pi}{w}y\right) \right]_0^w = 0 \\ b_n &= \frac{2}{w} \int_0^w \sin\left(n\frac{\pi}{w}y\right) dt = \frac{-2}{n\pi} \left[\cos\left(n\frac{\pi}{w}y\right) \right]_0^w = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\pi} & \text{for } n \text{ odd} \end{cases} \\ \Rightarrow 1 &= \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{1}{n} \sin\left(n\frac{\pi}{w}y\right) \end{aligned} \quad (6)$$

Naturally, 1 can also be expressed in an equivalent form in the z direction. This implies that both sides of (3) take on similar forms:

$$\begin{aligned} \frac{-\Delta p}{\eta L} &= \frac{-\Delta p}{\eta L} \cdot 1 \cdot 1 = \frac{-16}{\pi^2} \frac{\Delta p}{\eta L} \sum_{n \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \frac{1}{nm} \sin\left(n\frac{\pi}{w}y\right) \sin\left(m\frac{\pi}{h}z\right) \\ \nabla^2 u_x(y, z) &= -\pi^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \left(\frac{n^2}{w^2} + \frac{m^2}{h^2} \right) \sin\left(n\frac{\pi}{w}y\right) \sin\left(m\frac{\pi}{h}z\right) \end{aligned}$$

From this u_{nm} can be deduced, and it can be seen that only odd terms in (5) should be included:

$$u_{nm} = \frac{16}{\pi^4} \frac{\Delta p}{\eta L} \frac{1}{nm \left(\frac{n^2}{w^2} + \frac{m^2}{h^2} \right)} \quad (7)$$

$$u_x(y, z) = \frac{16}{\pi^4} \frac{\Delta p}{\eta L} \sum_{n \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \frac{1}{nm \left(\frac{n^2}{w^2} + \frac{m^2}{h^2} \right)} \sin\left(n\frac{\pi}{w}y\right) \sin\left(m\frac{\pi}{h}z\right) \quad (8)$$

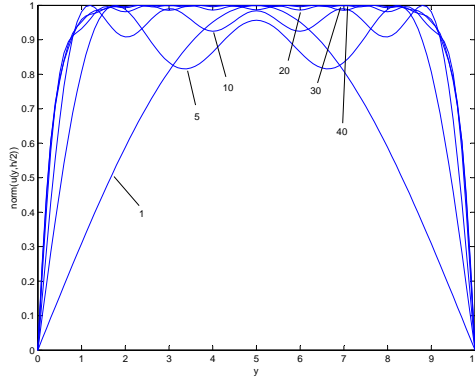
3.3 Poiseuille flow in a rectangular pipe – the velocity field

A plot of the solution (8) of the velocity field is wanted. To make this one have to find n, m values that are large enough for the series to reach the limit of convergence. Figure 2 shows test plots that can be used to determine the needed values for n and m . From these it is easily seen that the needed values grows rapidly for increasing $h : w$.

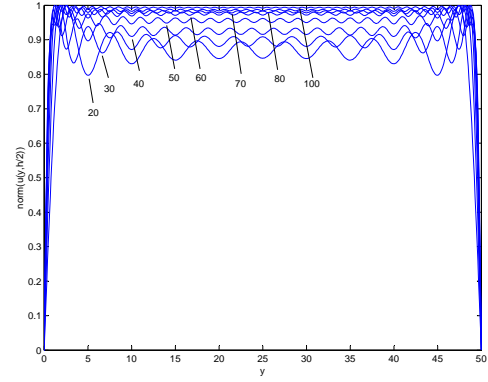
Naturally the n/m value affect only the resolution along the y/z direction, which means that one can minimise the number of calculations needed to compute (8); for instance $n \gtrsim 100$ is needed for $h : w = 1 : 50$ (figure 2.b) while only $m = 35$ is sufficient (figure 2.b). For simplicity $n = m$ will be used in the plots.

The plots in this section have been made with the aid of the MATLAB code in appendix A. Unless otherwise stated the amplitude of all plots is normalised to 1.

Contour and surface plots for $h : w = 1 : 1, 1 : 3, 1 : 10$ are shown in figure 3 to 5

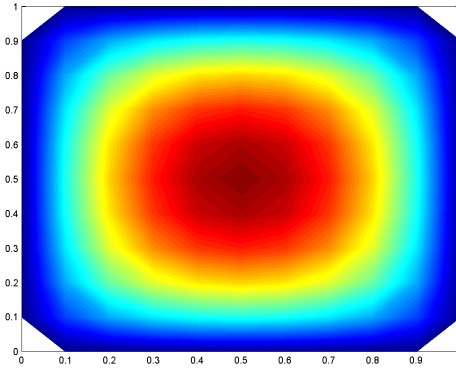


(a) 1:10 pipe.

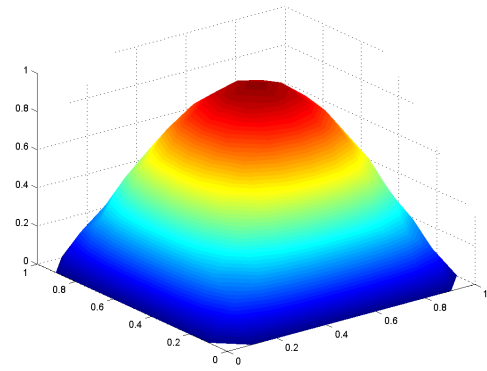


(b) 1:50 pipe.

Figure 2: Flow profile $u_x(y, h/2)$ for $h : w$ dimensions. For each plot the value of $n = m$ is shown.

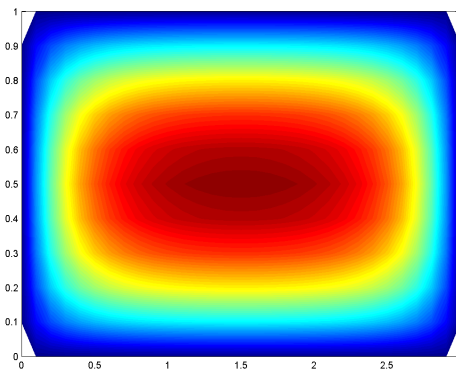


(a) Contour plot.

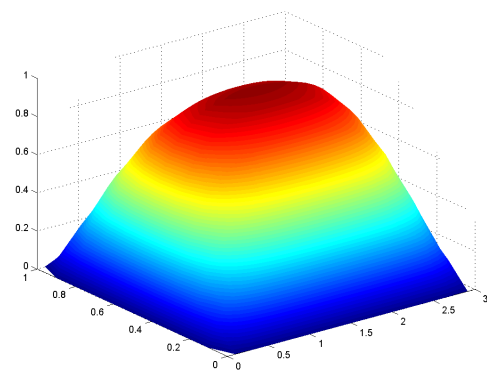


(b) Surface plot.

Figure 3: Flow profile $u_x(y, z)$ for $h : w = 1 : 1$ dimensions. $n = m = 35$ is used. If higher resolution were employed a circular inner structure would be seen.



(a) Contour plot.



(b) Surface plot.

Figure 4: Flow profile $u_x(y, z)$ for $h : w = 1 : 3$ dimensions. $n = m = 35$ is used. A more elliptic inner structure would be seen with increased resolution.

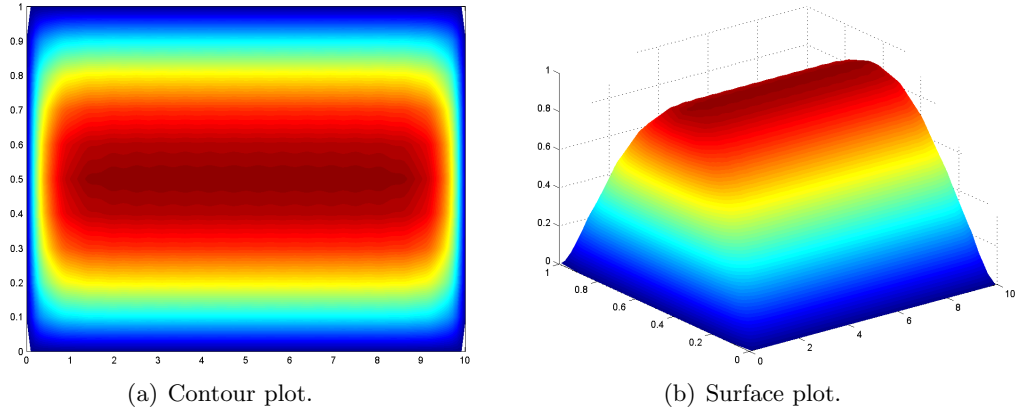


Figure 5: Flow profile $u_x(y, z)$ for $h : w = 1 : 10$ dimensions. $n = m = 35$ is used. Using higher values of n, m would remove the ripple.

Infinitely Wide Rectangular Pipe/Infinite Parallel Planes

The flow in the rectangular pipe can as seen in the plots be approximated with an infinitely wide rectangular pipe ($w \rightarrow \infty$) (IWRP) for increasing $h : w$. This can come in quite handy since computation of (8) can be quite time consuming. Following arguments similar to the ones used in section 3.1 one find the structure $\mathbf{U}(x, y, z) = (u_x(z), 0, 0)$. Checking that the continuity equation is obeyed, and that Navier-Stokes reduces to (3) is trivial. By checking the no-slip boundary condition, and inserting in (3) it can be shown that the velocity profile for the IWRP with $h = 2a$ is:

$$u_x^{IWRP}(z) = z(2a - z) \frac{\Delta p}{2\eta L} \quad (9)$$

This profile is plotted in figure 6.

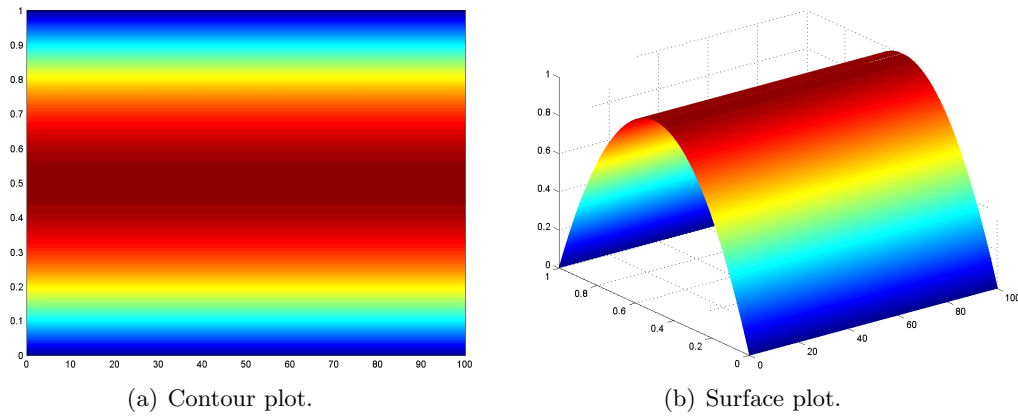


Figure 6: Flow profile $u_x(z)$ for IWRP.

How large $h : w$ relationship that is required to obtain a good approximation with the IWRP is addressed in the following. From figures 4, 5, 7, and 8 one can see that the main error along the y axis confines to $[0; a]$ and $[w - a; w]$ quite fast for increasing

$h : w$. This is easily understood from the $1 : 1$ case from where it is obvious that the wall affects the water to the a "depth". As an example the error for a $1 : 30$ geometry approximated with the IWRP is less than $1/30 = 3.3\%$.

A better error analysis can be performed by looking at the standard deviations between the actual flows and the flow in the IWRP. Since these calculations involves only one point from each flow the unbiased ($N - 1$ normalisation) standard deviation reduces to:

$$\begin{aligned}\sigma &= \sqrt{V} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \mu)^2} \quad , \quad \mu = \frac{1}{N} \sum_{i=1}^N x_i \\ &= \dots = \frac{1}{\sqrt{2}} |x_1 - x_2|\end{aligned}\tag{10}$$

Figure 7 and 8 shows the results of this calculation on different geometries. If one sum up the errors and normalise by the geometry $(h + 1)(w + 1)^*$ one find a number for how good the approximation is in each case; actually you get the average standard deviation in units of $\frac{\Delta p}{\eta L}$ since this factor is omitted in the calculation of both. The results of such summations are listed in table 1 where the mean value of u_x^{IWRP} is also stated. With this method one find an average error of each coordinate on $0.13148/7.5 = 1.75\%$ when the $1 : 30$ flow is compared to the IWRP flow.

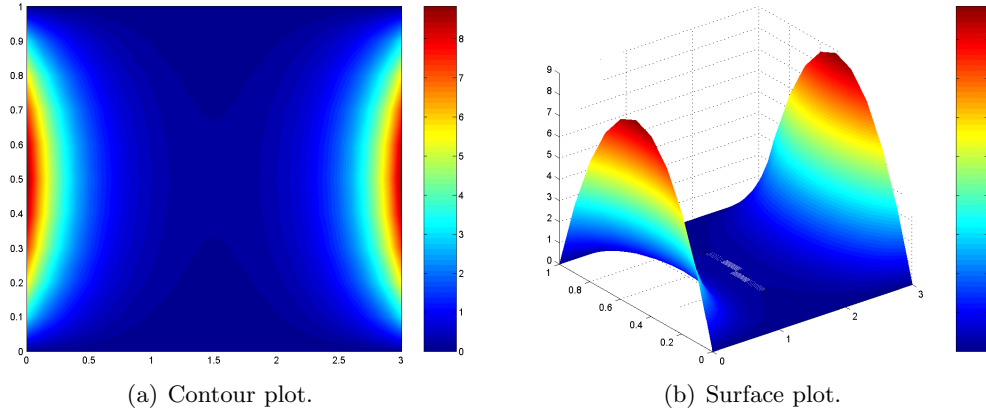
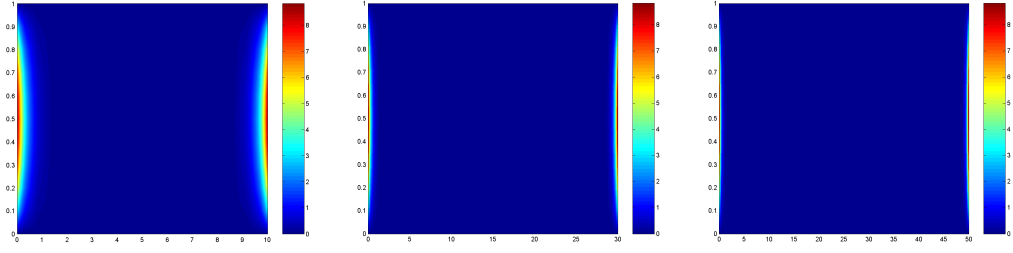


Figure 7: Standard deviation between IWRP and the $1 : 3$ pipe; $n = 35$. Units of $\frac{\Delta p}{\eta L}$.

The values of $\max(u_x^{finite})$ in table 1 might seem a bit odd since they exceed $\max(u_x^{IWRP}) = 12.5$. The higher velocities comes from the ripple from the finite approximation. This ripple can be removed by choosing larger values for n and m (e.g. try comparing the values for $1 : 30$ and $1 : 50$).

By plotting the average standard deviations ($stdavg$) from table 1 (figure 9.a) it is seen that there is a potential dependence of $\frac{w}{h}$. This is confirmed by the linear dependence one find by making a ln-ln plot (figure 9.b). An analytical expression for

*That 1 has to be added to each axis is due to the fact that a matrix containing u_x has rows and columns proportional to $(w + 1)$ and $(h + 1)$.



(a) Contour plot 1 : 10, $n = 35$. (b) Contour plot 1 : 30, $n = 120$. (c) Contour plot 1 : 50, $n = 120$.

Figure 8: Standard deviation between IWRP and pipes with different geometry. Units of $\frac{\Delta p}{\eta L}$.

	$h : w=1:3$	$h : w=1:10$	$h : w=1:30$	$h : w=1:50$
$\frac{1}{(h+1)(w+1)} \sum_{y=0}^w \sum_{z=0}^h \sigma_{yz}$	1.2595	0.39043	0.13148	0.083674
$\max(u_x^{finite})$	12.267	12.515	12.508	12.548
$\max(u_x^{IWRP})$	12.5	12.5	12.5	12.5

Table 1: Results from the error analysis. The mean value of u_x^{IWRP} is $\langle u_x^{IWRP} \rangle = 7.5$. Units of $\frac{\Delta p}{\eta L}$.

the standard deviation can easily be found:

$$\begin{aligned} \ln(stdavg) &= a \ln\left(\frac{w}{h}\right) + b \\ \Leftrightarrow stdavg &= \left(\frac{w}{h}\right)^a \cdot e^b \end{aligned} \quad (11)$$

By making the best linear fit $a = -0.96916$ and $b = 1.2911$ is found. A plot of the model is found in figure 9.

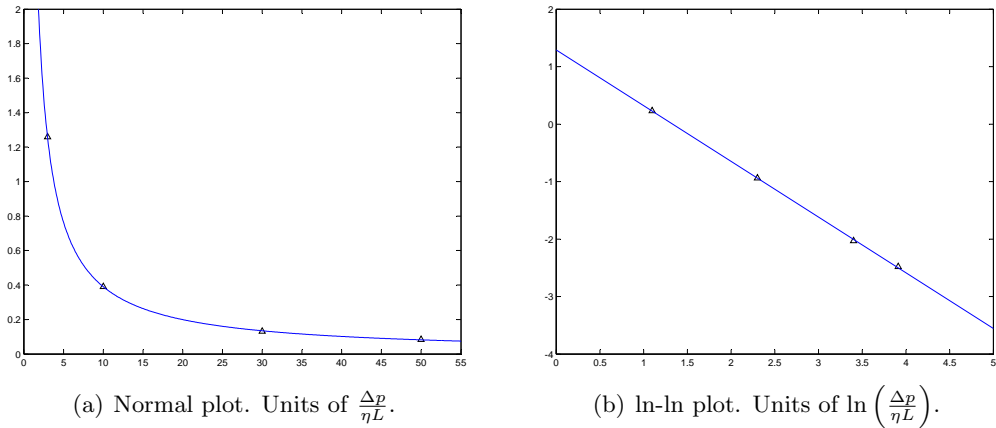


Figure 9: Average standard deviations as a function of $\frac{w}{h}$. Triangles show calculated points, and the curve shows the fitted model.

3.4 Poiseuille flow in a microfluidic laser

At MIC a microfluidic dye laser has been produced. The active part of it consists of a cavity (rectangular channel) where ethanol carrying the active dye is flowing. For proper functioning a flow rate of $Q = 10\mu\text{L}/\text{hour}$ is required. The length of the channel is $L = 122\text{mm}$, the width $w = 300\mu\text{m}$, and height $h = 10\mu\text{m}$. The viscosity of ethanol is close to water's – at room temperature it is $\eta_{\text{ethanol}} = 0.001197\text{Pa}\cdot\text{s}$.

In section 3.3 it was found that the error introduced when employing the IWRP for a 1 : 30 geometry is less than 2%. Therefore the IWRP is justifiably employed since the pressure drop needed to enable Q is more easily calculated with this model. The flow rate for the IWRP and the needed pressure drop is:

$$\begin{aligned} Q &= \int_0^w \int_0^h u_x^{IWRP}(z) \, dz \, dy = \int_0^w \int_0^h z(h-z) \frac{\Delta p}{2\eta L} \, dz \, dy \\ &= \frac{\Delta p}{2\eta L} \int_0^w \left[\frac{1}{2}hz^2 - \frac{1}{3}z^3 \right]_0^h \, dy = \frac{\Delta p}{12\eta L} \int_0^w h^3 \, dy \\ &= \frac{\Delta p}{\eta L} \frac{wh^3}{12} \\ \Leftrightarrow \Delta p &= \frac{12Q\eta L}{wh^3} = 1.6 \cdot 10^4 \text{Pa} \approx 0.16 \text{atm} \end{aligned} \tag{12}$$

This pressure drop is a reasonable Δp for a micro chip since it is quite low. It corresponds to the pressure of a 1.6m tall water column. By computing a more correct flow rate for $u_x^{finite}(y, z)$ one can find that the IWRP model can be improved by using $w - 0.6h$ instead of w .

3.5 The hydraulic resistance of a micro-mixer

The pressure drop is linked to the flow rate by the hydraulic resistance in a manner similar to Ohm's law for a electrical resistor:

$$\begin{aligned}\Delta p &= R_{hyd} Q \\ \Rightarrow R_{hyd}^{IWRP} &= \frac{\Delta p}{Q} = \frac{12\eta L}{wh^3}\end{aligned}\quad (13)$$

This expression can then be used to calculate the total hydraulic resistance of the micro-mixer shown in figure 10.a. The six inlet pressures are $p_0 + \Delta p$, and the outlet pressure is p_0 . Figure 10.b shows the equivalent circuit diagram for the mixer. The total resistance of the mixer (corners are assumed to have a zero resistance) is:

$$\begin{aligned}R_{inner} &= (R_1 + R_4) \parallel R_2 + R_4 \\ R_{branch} &= R_{inner} \parallel R_3 + R_5 \\ R_{tot} &= R_{branch} \parallel R_{branch} + R_5 = R_{branch}/2 + R_5\end{aligned}\quad (14)$$

R_1 - R_5 is calculated with (13) and R_{tot} is computed using the relations for serial and parallel resistors. For simplicity the IWRP model is used even though we only have a 1 : 6 geometry where the average standard deviation (11) is 8.5%. $\eta_{water} = 0.0010021928 \text{ Pa}\cdot\text{s}$ is used. The result of the calculations can be found in table 2.

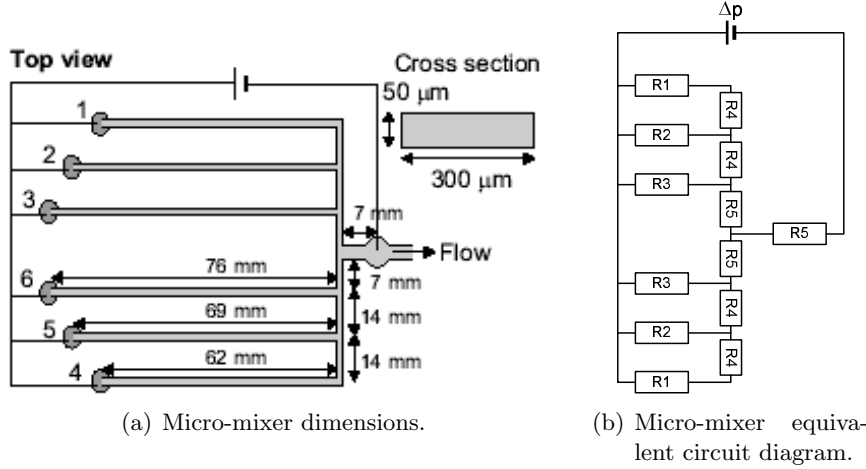


Figure 10: Drawing and model of the micro-mixer.

R_1	R_2	R_3	R_4	R_5	R_{inner}	R_{branch}	R_{tot}
$1.99 \cdot 10^{13}$	$2.21 \cdot 10^{13}$	$2.44 \cdot 10^{13}$	$4.49 \cdot 10^{12}$	$2.25 \cdot 10^{12}$	$1.61 \cdot 10^{13}$	$1.19 \cdot 10^{13}$	$8.21 \cdot 10^{12}$

Table 2: Hydraulic resistances. Units of $\frac{\text{Pa}\cdot\text{s}}{\text{m}^3}$.

A MATLAB Code

The MATLAB code listed in this section is not prepared for presentation! It is ugly and only intended to help the author remember what commands that was used to make the calculations in MATLAB for this assignment.

```
%Exercise 3
clear workspace;

% 1D plots for z=h/2 along y (actually it's the other way around, bet cuz of sym)

w = 10;
h = 10;
N = 1;
M = N;

u = zeros(w+1,h+1); %for 2D
%u = zeros(1,h+1); %for 1D

for y=0:w %for 2D
    %y=w/2; %for 1D
        for z=0:h
            for n=1:N
                for m=1:M
                    if (mod(n,2)+mod(m,2)==2) % only odd n and m's
                        u(y+1,z+1) = u(y+1,z+1) + 1/(n*m*(n^2/w^2 + m^2/h^2))*sin(n*pi*y/w)*sin(m*pi*z/h); %for 2D
                        %u(1,z+1) = u(1,z+1) + 1/(n*m*(n^2/w^2 + m^2/h^2))*sin(n*pi*y/w)*sin(m*pi*z/h); %for 1D
                    end
                end
            end
        end
    end
end %for 2D

u = u';
u = u/max(max(u)); % normalise

%1D plot
%Make multiple plots in same figure by running the script multiple times with varying m and n
%figure(10);
%plot(0:0.1:z/10,u);
%ylabel('norm(u(y,h/2))');
%xlabel('y');
%hold on;

%2D plots
%figure(1);
%contour(0:0.1:w/10, 0:0.1:z/10, u, 3000);
%%figure(2)
%%surf(0:0.1:w/10, 0:0.1:z/10, u); %the interpolation gives an arti-looking plot
%%shading interp;
%figure(3)
%contour3(0:0.1:w/10, 0:0.1:z/10, u, 3000);

%Infty rect pipe
clear workspace;
w = 1;
h = 50;
a = h/2;

u = zeros(h+1,w+1); %for 2D

for z=0:h
```

```

    u(z+1,1) = z*(2*a-z);
end

for i=1:w
    u(:,i+1) = u(:,1);
end

u = u/max(max(u)); % normalise

%2D plots
figure(1);
contourf(0:100:w*100, 0:0.1/5:z/50, u, 1000);
shading flat;
%figure(3)
%contour3(0:100:w*100, 0:0.1/5:z/50, u, 5000);

%Exercise 3 - Calc of avg std
clear workspace;
format long;
w = 300;
h = 10;
N = 120;
M = N;
u = zeros(w+1,h+1); %for 2D

for y=0:w %for 2D
    for z=0:h
        for n=1:N
            for m=1:M
                if (mod(n,2)+mod(m,2)==2) % only odd n and m's
                    u(y+1,z+1) = u(y+1,z+1) + 1/(n*m*(n^2/w^2 + m^2/h^2))*sin(n*pi*y/w)*sin(m*pi*z/h); %for 2D
                end
            end
        end
    end
end %for 2D
u = u';
max(max(u))
u = u*16/pi^4; %get in terms of Dp/(etaL)

%Infty rect pipe
a = h/2;
u2 = zeros(h+1,w+1); %for 2D

for z=0:h
    u2(z+1,1) = z*(2*a-z);
end

for i=1:w
    u2(:,i+1) = u2(:,1);
end
max(max(u2))
u2 = u2/2; %get in terms of Dp/(etaL)

error = 1/sqrt(2)*abs(u-u2);

% Exercise 3
% Plots for the error analysis
clear workspace;

errorX = [3 10 30 50];

```

```

errorY = [1.25949932267368 0.39042785855980 0.13148055121231 0.08367385765654];

figure(1);
plot(log(errorX),log(errorY),'k^');
hold on;
c = polyfit(log(errorX),log(errorY),1);
xpoints = [0 5]; %x points for the line
ypoints = polyval(c, xpoints);
plot(xpoints,ypoints,'b-');

figure(2); %points in normal plot plotted along with model
x = 0:0.1:55;
y = x.^c(1)*exp(c(2));
plot(errorX,errorY,'k^',x,y,'b-');
axis([0 55 0 2]);

%Exercise 3.5 - hydraulic resistances
clear workspace;

w = 300e-6;
h = 50e-6;
eta = 0.0010021928;

L = 7e-3

R = (12*eta*L) / (w*h^3)

R1 = 1.988350515199999e+013
R2 = 2.212841702399999e+013
R3 = 2.437332889599999e+013
R4 = 4.489823743999998e+012
R5 = 2.244911871999999e+012

R_inner = ((R1+R4)*R2) / (R1+R2+R4) + R4 %1.608816646002758e+013
R_branch = (R_inner*R3) / (R_inner+R3) + R5 %1.193615459581677e+013
R_tot = R_branch/2 + R5 %8.212989169908382e+012

```